

10/12/99

# MATHEMATICAL PROBLEMS IN THE CONTROL OF UNDERACTUATED SYSTEMS

DAVID AUCKLY

Department of Mathematics,  
Kansas State University, Manhattan, KS 66506-2602, USA  
email: dav@math.ksu.edu

LEV KAPITANSKI

Department of Mathematics,  
Kansas State University, Manhattan, KS 66506-2602, USA  
email: levkapit@math.ksu.edu

There are many interesting mathematical problems in control theory. In this paper we will discuss problems and techniques related to underactuated systems. An underactuated system is one with fewer control inputs than degrees of freedom. Balancing a ruler on the tip of a finger is a good example of an underactuated system. This system has five degrees of freedom (three for the fingertip and two angles for the ruler). However, only the three degrees of freedom for the fingertip are directly controlled. In fact, any system requiring balance is an underactuated system. A bicycle is an obvious example. An airplane is a less obvious example (six degrees of freedom, underactuated by two).

We will give a mathematical formulation of several problems arising from applications, review some standard and new techniques, and pose some interesting and challenging open questions.

## STABILIZATION OF UNDERACTUATED SYSTEMS

To describe a mechanical system we start with a manifold,  $Q$ , representing all possible configurations of the system. The configuration space  $Q$  is equipped with a Riemannian metric,  $g$ , so that the kinetic energy is  $\frac{1}{2}g(\dot{x}, \dot{x})$ . It also comes with a function  $V : Q \rightarrow \mathbb{R}$ ,

---

This work was partially supported by a grant from the National Science Foundation CMS 9813182.

and two fiber preserving maps  $c, f : TQ \rightarrow TQ$ . The function  $V$  represents potential energy,  $c$  represents dissipation, and  $f$  represents applied external forces. The equations of motion are given by

$$\nabla_{\dot{\gamma}} \dot{\gamma} + c(\dot{\gamma}) + \text{grad}_{\gamma} V = f(\dot{\gamma}). \quad (1)$$

The external forces,  $f$ , are used to control the system. The system obtained by setting the control input to zero is called *the open loop system*. Equation (1) including the control input is called *the closed loop system*. The system is *underactuated* if  $f$  is restricted to be 0 in some directions. In other words, there is a  $g$ -orthogonal projection  $P$  onto the subspace of unactuated directions, and  $P(f)$  must vanish.

The basic problem is to find a function  $f$  in some class so that solutions to equation (1) have some desired properties. Physically one may not always be able to measure the full state  $(x, \dot{x}) \in TQ$  of the mechanical system. In this situation the function  $f$  must only depend on the observable variables. For now, we will consider the case when all variables may be observed. This is referred to as full state feedback control.

The stabilization problem is to find a control input, so that some point  $(x_0, 0)$  will be an asymptotically stable equilibrium. Other notions of stability may also be considered, however, asymptotic stability is the most useful in applications. We will next review several approaches to the stabilization problem.

The most commonly employed technique used to address this problem is linearization. Choosing  $f$  so that the eigenvalues of the linearized equation lie in the left half plane, will ensure that the desired point is a locally asymptotically stable equilibrium. This reduces the problem to an algebraic question that can be easily solved and implemented.

First, note that it is not possible to stabilize every system, for example,

$$\begin{cases} \ddot{x}^1 - x^1 = 0 \\ \ddot{x}^2 + x^2 = u. \end{cases}$$

For a linear  $n \times n$  system of the form  $\dot{y} = Ay + Bu$  necessary and sufficient conditions for the existence of a linear stabilizing control law  $u = Cy$  are well known [14, 18]: the rank of the matrix  $[sI - A, B]$  must be  $n$  for all  $\text{Re } s \geq 0$ . If a linear stabilizing control law exists, there is a finite dimensional family of linear stabilizing control laws.

Once it is known that a stabilizing control law exists, one must choose a specific control law. The problem of finding a matrix  $C$  given the eigenvalues of  $A + BC$  is called pole placement. Engineers use various rules of thumb to decide where to place the poles.

These rules of thumb are based upon the behavior of solutions to a constant coefficient second order ODE. For higher order systems one purposefully places two dominant poles,  $z_1$  and  $z_2$ , with the remaining poles near the real axis and far to the left. This enables one to approximate solutions of the higher order system by solutions of a second order system.

#### EXAMPLE 1: THE INVERTED PENDULUM CART

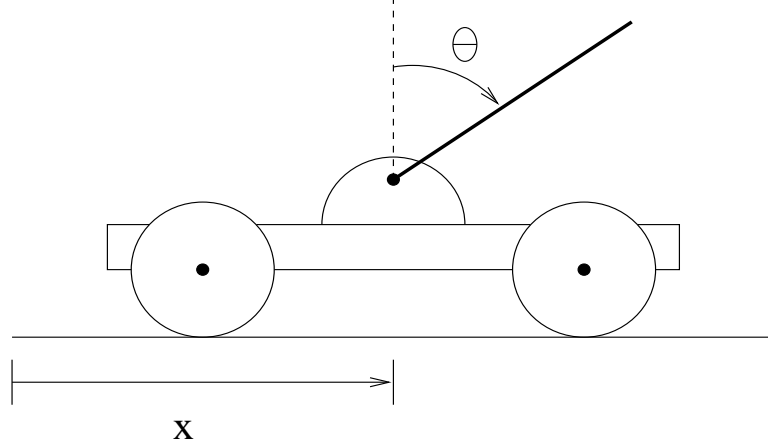


Figure 1

With appropriate scaling the metric  $g$  is given by  $g = d\theta^2 + 2b \cos(\theta) dx d\theta + dx^2$ , where  $b$  is a physical parameter,  $0 < b < 1$ . The potential energy is given by  $V = \cos(\theta)$ . Since no torques can be applied directly to the pendulum,  $P = (b \cos(\theta) dx + d\theta) \otimes \partial/\partial\theta$  is the orthogonal projection onto the direction  $\partial/\partial\theta$ . Assuming that there is no dissipation,  $c = 0$ . The  $\partial/\partial\theta$ - and  $\partial/\partial x$ -components of the equations of motion read:

$$\begin{aligned}\ddot{\theta} + b \cos(\theta) \ddot{x} - \sin(\theta) &= 0 \\ b \cos(\theta) \ddot{\theta} + \ddot{x} - b \sin(\theta) \dot{\theta}^2 &= u,\end{aligned}$$

where  $u = g(\partial/\partial x, f)$  represents the external force applied to the base of the cart, and it is the control input.

The linearization around  $\theta = 0$ ,  $x = 0$ ,  $\dot{\theta} = 0$ ,  $\dot{x} = 0$  reads

$$\frac{d}{dt} \begin{pmatrix} \theta \\ x \\ \dot{\theta} \\ \dot{x} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1-a_1b}{1-b^2} & \frac{-a_2b}{1-b^2} & \frac{-a_3b}{1-b^2} & \frac{-a_4b}{1-b^2} \\ \frac{a_1+b}{1-b^2} & \frac{a_2}{1-b^2} & \frac{a_3}{1-b^2} & \frac{a_4}{1-b^2} \end{pmatrix} \begin{pmatrix} \theta \\ x \\ \dot{\theta} \\ \dot{x} \end{pmatrix}$$

3

where  $u = a_1\theta + a_2x + a_3\dot{\theta} + a_4\dot{x}$  is the linearized control input. The characteristic polynomial of the above matrix is

$$\lambda^4 + \frac{a_3b - a_4}{1 - b^2}\lambda^3 + \frac{1 + a_2 - a_1b}{1 - b^2}\lambda^2 + \frac{a_4(1 + b^2)}{(1 - b^2)^2}\lambda + \frac{a_2(1 + b^2)}{(1 - b^2)^2}.$$

Once the desired eigenvalues are specified, it is an easy matter to solve for the  $a_k$  and get the control law.

Another standard technique employed in control design is linear quadratic optimal control, [18]. For linearized systems  $\dot{y} = Ay + Bu$ , one looks for control laws which will minimize the functional

$$J(y, u) = \int_0^\infty \langle y(t), Qy(t) \rangle + \langle u(t), Nu(t) \rangle dt$$

subject to the condition  $\dot{y} = Ay + Bu$ . Here  $Q$  and  $N$  are positive definite quadratic matrices which are usually specified by engineering rules of thumb.

If a stabilizing control law exists, the resulting control input can be expressed as a linear function of the state  $y$ .

Linearization works very well for many practical applications. Unfortunately, the limitations of linearization are seldomly discussed. Continuing in this tradition we will now address nonlinear methods without stating why.

Many nonlinear methods employ the notion of a Lyapunov function. For a system of ODEs,  $\dot{x} = f(x)$ , a Lyapunov function is a function  $F(x)$  which is bounded from below and decreases along the trajectories, [4]. It can be defined globally or locally. If  $x_0$  is a stationary solution of  $\dot{x} = f(x)$ , and a unique local minimizer of the Lyapunov function  $F$ , then  $x_0$  is Lyapunov stable. If  $F$  is nonconstant along nonstationary trajectories then  $x_0$  will be locally asymptotically stable. If, in addition,  $F$  is defined globally, then  $x_0$  is a global asymptotically stable equilibrium.

For the stabilization problem,  $\dot{x} = f(x, u)$ , one wishes to find a control input,  $u$ , as a function of the state,  $x$ , and a Lyapunov function,  $F(x)$ , so that the closed loop system admits  $F(x)$  as a Lyapunov function. In the linear case,  $\dot{y} = Ay + Bu$ , with  $u = Cy$ , one looks for a quadratic Lyapunov function  $F(y) = \langle y, Ky \rangle$ . The function  $F$  will be a Lyapunov function if and only if

$$D = (A + BC)^* K + K (A + BC) \quad (2)$$

is negative definite. Given any negative definite  $D$  one may solve equation (2) for  $K$  if and only if all the eigenvalues of  $A + BC$  are in the left half-plane. The solution is unique:

$$K = - \int_0^\infty e^{t(A+BC)^*} D e^{t(A+BC)} dt.$$

Note, that in the optimal control approach discussed previously, the value function

$$F(x) = \inf_u J(x, u)$$

is a Lyapunov function. We also remark that, if there is a locally asymptotically stabilizing control law, then there exists a local Lyapunov function, [4].

Several recent papers propose to find control inputs so that the closed-loop system (1) would have a natural candidate for a Lyapunov function, [1-3, 5-9, 13, 17]. In [3] we introduce the following approach to the stabilization problem for underactuated systems.

We consider the control problem

$$\nabla_{\dot{\gamma}} \dot{\gamma} + c(\dot{\gamma}) + \text{grad}_{\gamma} V = f(\dot{\gamma}), \quad (1)$$

subject to the constraint  $P(f) = 0$ . We wish to find  $f$  and a Lyapunov function simultaneously. In fact, we will describe an infinite dimensional family of control inputs.

Our approach to this question is to find functions  $\hat{g}$ ,  $\hat{c}$ ,  $\hat{V}$  and  $f$  so that solutions to Equation (1) are automatically solutions to

$$\widehat{\nabla}_{\dot{\gamma}} \dot{\gamma} + \widehat{c}(\dot{\gamma}) + \widehat{\text{grad}}_{\gamma} \hat{V} = 0. \quad (3)$$

This is the matching philosophy. The motivation for this philosophy is that

$$\hat{H}(X) = \frac{1}{2} \hat{g}(X, X) + \hat{V}$$

is a natural candidate for a Lyapunov function because  $\frac{d}{dt} \hat{H}(X) = -\hat{g}(\hat{c}(X), X)$ . A state,  $X_0 \in TQ$  will be an asymptotically stable equilibrium if  $\hat{H}(X) \geq 0$ , and  $-\hat{g}(\hat{c}(X), X) \geq 0$  with equality only at  $X_0$ .

Equations (1) and (3) clearly hold if and only if:

$$f(X) \equiv \nabla_X X - \widehat{\nabla}_X X + \text{grad}_{\gamma} V - \widehat{\text{grad}}_{\gamma} \hat{V} + c(X) - \hat{c}(X), \quad (4)$$

for every vector field  $X$ . The condition  $P(f) = 0$  then becomes a system of nonlinear partial differential equations for  $\widehat{g}$ ,  $\widehat{V}$ , and  $\widehat{c}$ . Notice that constant multiples of  $g$ ,  $c$ , and  $V$  satisfy  $P(f) = 0$  even when  $P$  has full rank. Thus, one would expect many solutions when  $P$  does not have full rank. Separating  $P(f) = 0$  into terms which are quadratic in the velocity, independent of the velocity or odd functions of the velocity gives:

$$P(\nabla_X X - \widehat{\nabla}_X X) = 0, \quad (5.1)$$

$$P(\text{grad}_\gamma V - \widehat{\text{grad}}_\gamma \widehat{V}) = 0, \quad (5.2)$$

$$P(c(X) - \widehat{c}(X)) = 0. \quad (5.3)$$

We will look for solutions to these matching equations with  $\widehat{g}$  non-degenerate so that  $g(X, Y) = \widehat{g}(\lambda X, Y)$  with  $\lambda \in \Gamma(T^*Q \otimes TQ)$ . It is clear that  $\lambda$  has to be  $g$  self-adjoint, i.e.,  $g(\lambda X, Y) = g(X, \lambda Y)$ . We will derive a linear system of partial differential equations for  $\lambda$  which must be satisfied if  $\widehat{g}$  is to solve Equation (5.1).

In our previous paper, we described a method to find every solution to the matching equations by solving three linear systems of partial differential equations in a row. We will review this method now.

One first solves the equations

$$\nabla g \lambda|_{\text{Im } P \otimes 2} = 0, \quad (6)$$

for  $\lambda|_{\text{Im } P}$ . Then one solves

$$L_{\lambda P X} \widehat{g} = L_{P X} g \quad (7)$$

(this is a slight rewrite of equation (1.12) of our previous paper [3]),

$$L_{\lambda P X} \widehat{V} = L_{P X} V \quad (8)$$

(this is equation (1.13) of our previous paper [3]), then after solving equation (5.3), the control input will be given by (4).

In the previous paper we explicitly showed that any solution to the matching equations solves equations (6), (7), (8), and (5.3) (Propositions 1.1, 1.2, and 1.3 of [3]). Implicit in [3, Proposition 1.4] is the fact that any solution of equations (6), (7), (8) and (5.3) is in turn a solution to the matching equation. We will make this argument explicit now.

Indeed, taking into account the fact that  $P$  is  $g$ -selfadjoint, our  $\widehat{g}$ -Equation (7) implies the matching equation (5.1). Here is a short proof. For any  $Z$  and  $X$  we have

$$\begin{aligned}
& g(P(\widehat{\nabla}_Z Z - \nabla_Z Z), X) \\
&= g(\widehat{\nabla}_Z Z - \nabla_Z Z, PX) \\
&= \widehat{g}(\widehat{\nabla}_Z Z, \lambda PX) - g(\nabla_Z Z, PX) \\
&= Z\widehat{g}(Z, \lambda PX) - \widehat{g}(Z, \widehat{\nabla}_Z \lambda PX) - Zg(Z, PX) + g(Z, \nabla_Z PX) \\
&= -\widehat{g}(Z, \widehat{\nabla}_{\lambda PX} Z) + \widehat{g}(Z, [\lambda PX, Z]) + g(Z, \nabla_{PX} Z) - g(Z, [PX, Z]) \\
&= -\frac{1}{2}(L_{\lambda PX} \widehat{g}(Z, Z)) + \frac{1}{2}(L_{PX} g(Z, Z)) \\
&= 0.
\end{aligned}$$

Since this is true for all  $X$ ,  $P(\widehat{\nabla}_Z Z - \nabla_Z Z) = 0$ .

Equations (6) and (7) imply additional compatibility conditions. Even though we do not know all the compatibility conditions in general, we do know all the compatibility conditions for systems with two degrees of freedom. Let us summarize our method in the case of two degrees of freedom one of which is unactuated. Since the unactuated subspace is one dimensional, it can be locally expressed as the span of a unit length vectorfield,  $PX$ . Choose coordinates  $x^1, x^2$  so that  $PX = \frac{\partial}{\partial x^1}$ . In these coordinates  $g_{11} = 1$ . We will always write  $\lambda PX = \sigma \frac{\partial}{\partial x^1} + \mu \frac{\partial}{\partial x^2}$ , where  $\sigma$  and  $\mu$  are yet to be found. The  $\lambda$ -equation may be rewritten as

$$\frac{\partial}{\partial x^1}(g_{11}\sigma + g_{12}\mu) - 2[11, 2]\mu = 0, \quad \frac{\partial}{\partial x^2}(g_{11}\sigma + g_{12}\mu) - 2[12, 2]\mu = 0. \quad (9)$$

Here,

$$[ij, k] = g(\nabla_{\partial_i} \partial_j, \partial_k) = \frac{1}{2}(g_{ik,j} + g_{jk,i} - g_{ij,k}). \quad (10)$$

For these equations to be consistent the following compatibility condition must hold:

$$\frac{\partial}{\partial x^2}([11, 2]\mu) = \frac{\partial}{\partial x^1}([12, 2]\mu). \quad (11)$$

Notice that this is a first order partial differential equation. Theoretically, it can be solved for  $\mu$  via the method of characteristics. Generically, a solution will include an arbitrary function of a single variable. Once  $\mu$  is known,  $\sigma$  is given by

$$\sigma(x^1, x^2) = g_{12}(x^1, x^2)\mu(x^1, x^2) + 2 \int ([11, 2]\mu(x^1, x^2) dx^1 + [12, 2]\mu(x^1, x^2) dx^2) .$$

The next step is to solve equations (6) for  $\widehat{g}$ . It turns out that it is easiest to solve first for  $\widehat{g}_{11}$  and then find the remaining components from the algebraic system

$$g = \widehat{g}\lambda.$$

First note that the  $\{11\}$  component of the right side of (7) is

$$(L_{PX}g)(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^1}) = \frac{\partial}{\partial x^1}g_{11} = 0.$$

Next,

$$(L_{\lambda PX}\widehat{g})(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^1}) = \lambda PX(\widehat{g}_{11}) - 2\widehat{g}([\lambda PX, \frac{\partial}{\partial x^1}], \frac{\partial}{\partial x^1}).$$

Since

$$[\lambda PX, \frac{\partial}{\partial x^1}] = -\frac{\partial\sigma}{\partial x^1}\frac{\partial}{\partial x^1} - \frac{\partial\mu}{\partial x^1}\frac{\partial}{\partial x^2} = (-\frac{\partial\sigma}{\partial x^1}\frac{\partial}{\partial x^1} + \frac{\sigma}{\mu}\frac{\partial\mu}{\partial x^1})\frac{\partial}{\partial x^1} - \frac{1}{\mu}\frac{\partial\mu}{\partial x^1}\lambda PX$$

and  $\widehat{g}\lambda = g$ , we obtain,

$$\lambda PX\widehat{g}_{11} - 2(-\frac{\partial\sigma}{\partial x^1} + \frac{\sigma}{\mu}\frac{\partial\mu}{\partial x^1})\widehat{g}_{11} + 2\frac{1}{\mu}\frac{\partial\mu}{\partial x^1}g_{11} = 0.$$

Thus,  $\widehat{g}_{11}$  satisfies the following first order partial differential equation

$$\sigma\frac{\partial\widehat{g}_{11}}{\partial x^1} + \mu\frac{\partial\widehat{g}_{11}}{\partial x^2} + 2(\frac{\partial\sigma}{\partial x^1} - \frac{\sigma}{\mu}\frac{\partial\mu}{\partial x^1})\widehat{g}_{11} + 2\frac{\frac{\partial\mu}{\partial x^1}}{\mu} = 0.$$

The general solution to this first order PDE has an arbitrary function of a single variable in it. It is, once again, possible to solve this equation by the method of characteristics. Denote by  $y(x^1, x^2)$  any solution of the homogeneous equation

$$\sigma\frac{\partial y}{\partial x^1} + \mu\frac{\partial y}{\partial x^2} = 0 \tag{12}$$

such that  $\partial y/\partial x^2 \neq 0$ . Let  $\bar{\sigma}$  and  $\bar{\mu}$  be  $\sigma$  and  $\mu$  considered as functions of  $x^1$  and  $y$ , i.e.,

$$\bar{\sigma}(x^1, y(x^1, x^2)) = \sigma(x^1, x^2), \quad \bar{\mu}(x^1, y(x^1, x^2)) = \mu(x^1, x^2).$$

Then the solution to equation (7) is given explicitly by

$$\widehat{g}_{11}(x^1, x^2) = \frac{\mu^2}{\sigma^2} \left[ -2 \int_0^{x^1} \frac{\bar{\sigma}}{\bar{\mu}^3} \frac{\partial \bar{\mu}}{\partial x^1} dx^1 \Big|_{y=y(x^1, x^2)} + h(y(x^1, x^2)) \right], \tag{13}$$



where  $h(y)$  is an arbitrary function of a single variable. After  $\widehat{g}_{11}$  is found, we have

$$\widehat{g}_{12} = \frac{1}{\mu}(g_{11} - \sigma\widehat{g}_{11}), \quad \widehat{g}_{22} = \frac{1}{\mu}(g_{12} - \sigma\widehat{g}_{12}). \quad (14)$$

The equation for  $\widehat{V}$  reads

$$\sigma \frac{\partial \widehat{V}}{\partial x^1} + \mu \frac{\partial \widehat{V}}{\partial x^2} = \frac{\partial V}{\partial x^1}.$$

Again, this equation can be solved by the method of characteristics. Once a solution  $y$  of (12) is known,  $\widehat{V}$  is given by:

$$\widehat{V}(x^1, x^2) = - \int_0^{x^1} \frac{\bar{V}_{x^1}}{\bar{\sigma}} dx^1 \Big|_{y=y(x^1, x^2)} + w(y(x^1, x^2)), \quad (15)$$

where  $\bar{V}_{x^1}(x^1, y(x^1, x^2)) = \frac{\partial V(x^1, x^2)}{\partial x^1}$ . Also,

$$\widehat{c}_1 = c_1 + g_{12}(\widehat{c}_2 - c_2).$$

It is convenient to write the non-zero control input as a sum:

$$u = g(f, \frac{\partial}{\partial x^2}) = g(\nabla_X X - \widehat{\nabla}_X X + \widehat{grad}_\gamma V - \widehat{grad}_\gamma \widehat{V} + c(X) - \widehat{c}(X), \frac{\partial}{\partial x^2}) = u_g + u_V + u_c, \quad (16)$$

where

$$u_g = g(\nabla_X X - \widehat{\nabla}_X X, \frac{\partial}{\partial x^2}), \quad u_V = g(\widehat{grad}_\gamma V - \widehat{grad}_\gamma \widehat{V}, \frac{\partial}{\partial x^2}), \quad u_c = g(c(X) - \widehat{c}(X), \frac{\partial}{\partial x^2}),$$

In coordinates,

$$u_g = ([ij, 2] - g_{k2} \widehat{\Gamma}_{ij}^k) \dot{x}^i \dot{x}^j, \quad (17)$$

where  $\widehat{\Gamma}_{ij}^k$  are the Christoffel symbols corresponding to  $\widehat{g}$ ,

$$\widehat{\Gamma}_{ij}^k = \widehat{g}^{kp} [\widehat{ij}, p],$$

$\widehat{g}^{kp}$  is the inverse matrix to  $\widehat{g}_{lm}$ , and  $[\widehat{ij}, p]$  is defined as in (9) with all  $g$ 's replaced by  $\widehat{g}$ 's. The next term is:

$$u_V = V_{x^2} - \widehat{g}^{ij} g_{j2} \widehat{V}_{x^i}. \quad (18)$$

Finally,

$$u_c = \det g (c_2 - \widehat{c}_2). \quad (19)$$

From the explicit formulae (16)-(19) one sees that the first order germs of our control inputs contain every possible linear control law:

**Theorem.** *If a system with 2 degrees of freedom underactuated by 1 is linearly stabilizable, then, by choosing appropriate solutions of our matching equations, the linearization of the controlled system will have prescribed eigenvalues in the left halfplane. Even if the system is not stabilizable, any linear control law may be obtained as the first order germ of some control law in our family.*

## OPEN PROBLEMS

One of the reasons that the limitations of linearization are seldomly discussed is that there is no effective criteria to compare arbitrary control laws. To be more specific we will restrict our discussion to the stabilization problem.

A good stabilizing control law will produce a large basin of attraction, send solutions to the equilibrium in a short period of time, and will have low cost. Of course, the precise meaning of “large”, “short”, and “low” depends on the concrete engineering problem. It is not clear how to quantify these concepts. For linear systems the size of the basin of attraction is irrelevant since the whole space is the basin of attraction of a stable equilibrium. For nonlinear systems this question is subtle. One could just use the volume or diameter as a measure of the size of the basin of attraction. These are not, however, usually appropriate measures of size, see Figure 2. In addition, they are difficult to compute.

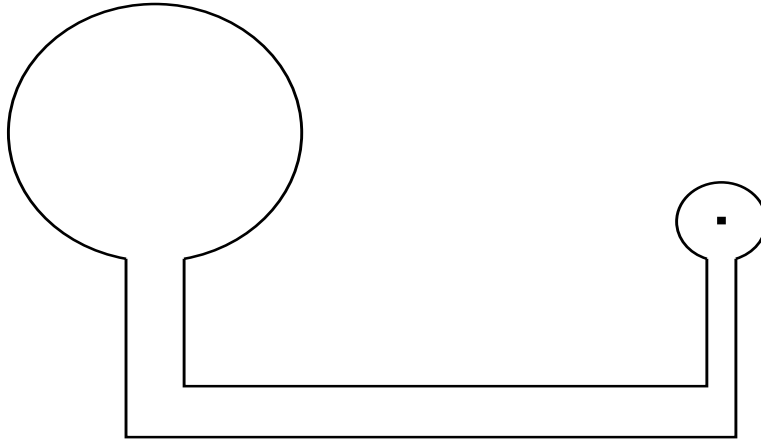


Figure 2

Alternatively, one could measure the radius of the largest inscribed ball centered at the equilibrium. It is not, however, clear which metric should be used in state space. It

is difficult to analytically or numerically estimate this radius. Even if one could compute this radius, it might not be the most relevant measure of performance. Real systems have an operating range: rollerblades are not designed to handle Mach 2. So, let  $\mathcal{B}$  be the basin of attraction and  $\mathcal{O}$  be the operating range. It is more reasonable to measure the size of the subset  $\mathcal{N} \subset \mathcal{B}$  consisting of all states whose forward trajectories remain in the operating range, see Figure 3. We will call  $\mathcal{N}$  *the normal operating range*. It is just as difficult to form a reasonable measure of the size of  $\mathcal{N}$ .

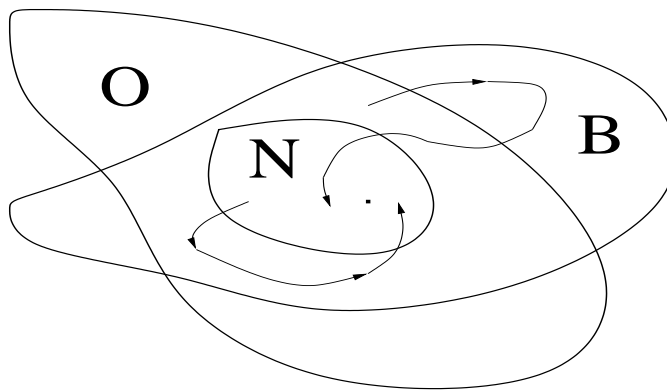


Figure 3

**Problem.** Find an effective analytical or numerical method to estimate the injectivity radius of the normal operating range or define a better measure of “size” together with an effective analytical or numerical estimate.

In engineering applications the time it takes to drive the states to the equilibrium is very important. This is usually characterized by the rise time, peak time, settling time, etc., [11, p. 222]. These notions are only well defined for solutions of  $m\ddot{z} + c\dot{z} + kz = f$  with  $m$ ,  $c$ ,  $k$ , and  $f$  constants. Considering the initial conditions with  $\dot{z} = 0$  only, one rescales the initial value problem to  $\ddot{z} + 2\zeta\dot{z} + z = 1$ ,  $z(0) = 0$ ,  $\dot{z}(0) = 0$ . The rise time,  $T_r$ , is the first time  $z(t)$  reaches 1. The time constant is  $1/\zeta$ . The settling time is the time it takes  $z(t)$  to get and stay within 2% of the final value 1. These notions are used for linear constant coefficient ODEs which have two dominant poles.

One can define analogues of these notions for nonlinear systems. An engineer may choose a *target operating range*  $\mathcal{D}$ , which is a small neighborhood of the equilibrium.

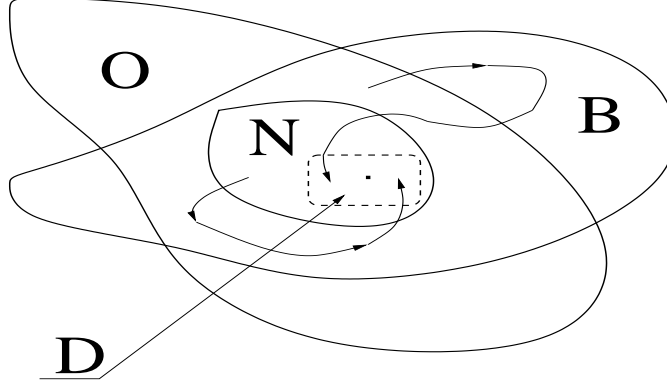


Figure 4

Define the  $\mathcal{ND}$ -settling time

$$T_{\mathcal{ND}} = \sup_{x_0 \in \mathcal{N}} \{t \geq 0 \mid x(t; x_0) \notin \mathcal{D}\}.$$

This is the time after which any trajectory starting in the normal operating range  $\mathcal{N}$  will settle into the target range  $\mathcal{D}$ .

**Problem.** *Find an effective analytical or numerical method to estimate the  $\mathcal{ND}$ -settling time or define a better measure of “short time” together with an effective analytical or numerical estimate.*

Another important characteristic of a control law is the cost. One often wishes to minimize some function of the trajectory and/or the control input (for example, minimize the energy used to complete a task). This imposes a non-trivial restriction on a control problem. Real life systems have additional physical restrictions. For example, there is a maximal voltage which may be applied to a DC motor before it saturates.

Reiterating, a good stabilizing control law should maximize the basin of attraction and minimize the time and cost while operating within the physical restrictions of the given system.

The problem of maximizing the size of the basin of attraction taken to the extreme leads to the question of finding the topologically best control laws. For example, the state space for the inverted pendulum cart is  $S^1 \times \mathbb{R} \times \mathbb{R}^2$ . For topological reasons it is impossible to find a control law so that the resulting flow has a globally asymptotically stable equilibrium in this case. The best one could hope for is a flow with a regular compact global attractor, [12, 16]. Recall that such an attractor is the union of the

unstable manifolds of finitely many hyperbolic fixed points. Also recall that the shape (in the sense of K. Borsuk) of the (global) attractor must be the same as the shape of the state space, [15]. We wish to minimize the number of fixed points. For the inverted pendulum cart *the topologically best flow* is depicted in Figure 5.

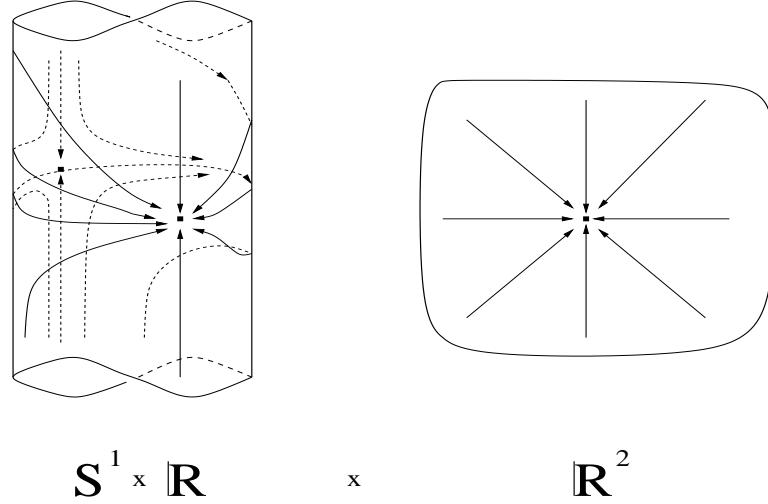


Figure 5

This flow must have one index 0 fixed point and one index 1 fixed point. The basin of attraction of the index 0 fixed point is the complement of a properly embedded  $\mathbb{R}^3$ .

**Problem.** *When is it possible to find a control input producing a topologically best flow? Is there a method to find such control laws?*

At the moment we do not even know whether a topologically best control law exists for the inverted pendulum cart. Furthermore, we do not know whether a control law with a global attractor exists!

A related problem is to describe the behavior of the system far away from the equilibrium. It is possible to compactify some dynamical systems with algebraic nonlinearities, [10]. The advantage of a compactification is that overall dynamics may be well understood by linearization about each of the critical submanifolds. The problem is that for nonalgebraic nonlinearities it is not known when a compactification exists. It is even interesting and instructive to try to compactify the system  $\ddot{\theta} + c\dot{\theta} + \sin(\theta) = 0$ .

We wrote this paper to encourage other people to think about nonlinear control theory. It has a great many unanswered fundamental questions. The field is ready for new ideas.

## REFERENCES

1. D. Auckly, L. Kapitanski, A. Kelkar, W. White, *Matching and pole placement for underactuated systems*, Preprint (1999).
2. D. Auckly, L. Kapitanski, A. Kelkar, W. White, *Matching and digital control implementation for underactuated systems*, Preprint (1999).
3. D. Auckly, L. Kapitanski, W. White, *Control of nonlinear underactuated systems*, to appear in Communications on Pure Appl. Math. (1999).
4. N. P. Bhatia and G. P. Szegő, *Dynamical systems: stability theory and applications*, Lecture Notes in Mathematics, 35, Springer-Verlag, Berlin Heidelberg New York, 1967.
5. A. Bloch, N. Leonard and J. Marsden, *Stabilization of mechanical systems using controlled Lagrangians*, Proc. IEEE Conf. Dec. Contr., San Diego, Calif., 1997, pp. 2356-2361.
6. A. Bloch, N. Leonard and J. Marsden, *Matching and stabilization by the method of controlled Lagrangians*, Proc. IEEE Conf. Dec. Contr., Tampa, Fla., 1998, pp. 1446-1451.
7. A. Bloch, N. Leonard and J. Marsden, *Stabilization of the pendulum on a rotor arm by the method of controlled Lagrangians*, Proc. IEEE Int. Conf. on Robotics and Automation, Detroit, Mich., 1999, pp. 500-505.
8. A. Bloch, N. Leonard and J. Marsden, *Controlled Lagrangians and a stabilization of mechanical systems I: The first matching theorem*, Preprint (1999).
9. A. Bloch, N. Leonard and J. Marsden, *Potential shaping and the method of controlled Lagrangians*, Preprint (1999).
10. O. I. Bogoyavlensky, *Methods in the qualitative theory of dynamical systems in astrophysics and gas dynamics*, Springer Series in Soviet Mathematics, Springer-Verlag, Berlin-New York, 1985.
11. R. C. Dorf and R. H. Bishop, *Modern control systems*, Addison-Wesley, 1995.
12. J. K. Hale, *Asymptotic Behavior of Dissipative Systems*, Mathematical Surveys and Monographs, vol. 25, Amer. Math. Soc., Providence, RI, 1987.
13. J. Hamberg, *General matching conditions in the theory of controlled Lagrangians*, to appear in proceedings of the 38th Conference on Decision and Control, Phoenix, Ariz., 1999.
14. T. Kailath, *Linear systems*, Prentice-Hall, Englewood Cliffs, N.J., 1980.

15. L. Kapitanski and I. Rodnianski, *Shape and Morse theory of attractors*, Communications in Pure Appl. Math. (1999) (to appear).
16. O. A. Ladyzhenskaya, *Attractors for Semigroups and Evolution Equations*, (Lezioni Lincee), Cambridge University Press, Cambridge, 1991.
17. A. J. van der Schaft, *Stabilization of Hamiltonian systems*, Nonlinear Analysis, Theory, Methods & Applications **10** (1986), no. 10, 1021-1035.
18. E. D. Sontag, *Mathematical control theory*, Texts in Applied Mathematics, 6, Springer-Verlag, New York, 1990.